Record-breaking statistics for random walks in the presence of measurement error and noise

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We address the question of distance record-setting by a random walker in the presence of measurement error, δ, and additive noise, γ and show that the mean number of (upper) records up to n steps still grows universally as $\langle R_n \rangle \sim n^{1/2}$ for large n for all jump distributions, including Lévy flights, and for all δ and γ. In contrast to the universal growth exponent of $1/2$, the pace of record setting, measured by the pre-factor of $n^{1/2}$, depends on δ and γ. In the absence of noise ($\gamma = 0$), the pre-factor $S(\delta)$ is evaluated explicitly for arbitrary jump distributions and it decreases monotonically with increasing δ whereas, in case of perfect measurement $(\delta = 0)$, the corresponding pre-factor $T(\gamma)$ increases with γ. Our analytical results are supported by extensive numerical simulations and qualitatively similar results are found in two and three dimensions.

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An upper record (record, for short) occurs at step n in a time series if the n-th entry exceeds all previous entries. The statistics of record breaking events in a discrete-time series with independent and identically distributed (i.i.d) entries have been studied in statistics and mathematics literature for a long time [1–3]. Recent years have seen a resurgence of interest in record statistics, which play a major role in the analysis of time series in a number of diverse contexts, including sports [4–7], biological evolution models [8, 9], theory of spin-glasses [10, 11], models of growing networks[12], analysis of climate data [13–17], and quantum chaos [18]. The quantity of central interest is the mean number of records $\langle R_n \rangle$ up to step n. For a time series with i.i.d entries, a striking universal result is that $\langle R_n \rangle \sim \ln n$ for large n [1], independent of the distribution of the individual entries. However, this universal logarithmic growth breaks down when the entries of the time series are *strongly correlated*, the simplest example being the case of a random walk where the entries of the time series represent the positions of the walker at discrete time steps.

While the subject of random walks has an enormous range of applications well beyond the original context of diffusion and Brownian motion, its exploration in terms of record setting is relatively recent. The basic question is: how often does a random walker, moving in continuous space by jumping a random distance at each discrete time step, set a distance record, i.e., advance farther from the origin than at all prior steps? In other words, how does the mean number of such record-setting events grow with the number of steps? This is a very natural question in many different contexts, such as in the evolution of stock prices [19, 20] or in queueing theory [21]. In the one-dimensional case, with pure diffusion but no overall drift, a universally valid result was found in [22] for the mean of the upper record-setting events $\langle R_n \rangle$, namely, that it scales as $(2/\sqrt{\pi})\, n^{1/2}$ for large n, where n is the number of steps, regardless of the length distribution of jumps (e.g., holds even for Lévy flights). This square root growth of the mean record number was also found numerically in two and three dimensions, and considering a drift, an abrupt shift in the scaling exponent from $1/2$ to 1 was identified [23]. Exact analytical results were also found in one dimension for a random walker with arbitrary drift [24, 25], for a continuous time random walker [26] and for multiple random walkers [27]. In the latter case, the theoretical results were in good agreement with an analysis of multiple stocks from the Standard & Poors 500 index [27].

However, to apply the above results to interpretation of *real* experiments, one needs to re-examine the notion of a record because the phrase "advance farther from the origin than at all prior time steps" requires closer examination. Why? Because all *real* measurements involve instrument error, δ, and noise, γ, is unavoidable. For instance, δ can be the assurance limit of the detector while γ can describe white noise from an instrument reading. Ties become possible because of the "fuzziness", as discussed, for example, in [17, 28, 29]. Hence, the question arises: how does the presence of measurement error δ or noise γ affect the growth of record number and the associated record-setting pace? Related questions were raised in the statistics literature, e.g., in terms of δ-exceedance records [30, 31] and very recently in the physics literature [29], but asymptotic results are available only for time series with i.i.d entries. To the best of our knowledge, the question has never been raised in the context of correlated entries such as random walks. For example, does the $\langle R_n \rangle \sim n^{1/2}$ scaling persist despite the presence of δ or γ and for various jump length distributions? If so, how is the pre-factor affected? We address these questions here using exact calculations and detailed Monte Carlo simulations. As a preview, our major finding is the decoupling between the growth exponent which remains universal and the pre-factor which carries the "burden" of finite precision and noise.

Rather than working with absolute values, we define a "one-sided" record (only positive maxima are considered) so that the i-th entry in a time series, x_i, is a record-breaking event (record, for short) if it exceeds all previous values in the sequence, i.e., if $x_i >$ max $(x_1, x_2, \ldots, x_{i-1})$. We shall hence-

forth interpret x_i as the distance of the random walker from the origin at the i-th time step. However, because of the presence of (a fixed) measurement error, δ, we shall now define x_i to be a record-breaking event (δ-record, for short) only if it exceeds all previous values in the sequence by, *at least*, δ. Similarly, accounting for measurement noise, x_i is a record-breaking event if, with the addition of γ (white noise), it exceeds all previous values in the sequence. A subtlety is that in the presence of error, a record can be defined as being larger – by the amount of the error – than the last record, or than the last maximum, the two being identical in the absence of error. In the analysis below, we enumerate records larger than the previous maximum, as it is more amenable to theoretical development as we show below.

We focus first on the influence of measurement error δ. Consider a discrete-time sequence $\{x_0 = 0, x_1, x_2, \dots, \}$, representing the position of a one-dimensional random walker starting at the origin $x_0 = 0$. The position x_m at step m is a continuous stochastic variable that evolves via the Markov rule, $x_m = x_{m-1} + \eta_m$ where η_m represents the jump at step m. The noise variables η_m's are independent and identically distributed random variables, each drawn from a symmetric and continuous jump density $f(\eta)$. Note that although η_m's are uncorrelated, x_m's are correlated random variables. We are interested in the statistics of the number of records R_n up to step n. A record occurs at step m if $x_m - \delta \geq x_k$ for all $k = 0, 1, 2, \dots, (m-1)$ where $\delta \geq 0$ represents the measurement error. For $\delta = 0$, the statistics of R_n are known to be universal, i.e., independent of the jump density $f(\eta)$ [22]. For instance, the mean record number $\langle R_n \rangle$ up to step n is given by the expression [22]

$$\langle R_n \rangle = (2n+1) \binom{2n}{n} 2^{-2n} \xrightarrow[n\to\infty]{} \frac{2}{\pi^{1/2}}\, n^{1/2} \,. \quad (1)$$

We now proceed to examine how $\langle R_n \rangle$ is affected by the measurement error δ. Define an indicator $\sigma_m = \{1, 0\}$ at each step m so that $\sigma_m = 1$ if a record occurs at step m and is 0 otherwise. We call $x_0 = 0$ a record, i.e., $\sigma_0 = 1$. Then evidently the number of records R_n up to step n in a given sequence is $R_n = \sum_{m=0}^{n} \sigma_m$. Next, we average this expression over different histories. Because σ_m is a binary $\{1, 0\}$ variable, its average $\langle \sigma_m \rangle$ is just the probability that a record occurs at step m. Hence,

$$\langle R_n \rangle = \sum_{m=0}^{n} \langle \sigma_m \rangle = \sum_{m=0}^{n} r_m(\delta), \quad (2)$$

where $r_m(\delta)$ denotes the record rate, i.e., the probability that a record occurs at step m. By definition, $r_0 = 1$. Hence,

$$r_m(\delta) = \mathrm{Prob}\left[x_m - \delta \geq \max\left[0, x_1, x_2, \dots, x_{m-1}\right]\right] \,. \quad (3)$$

Thus, $r_m(\delta)$ is the probability of the event that the random walker, starting at the origin, reaches x_m at step m, while staying below $x_m - \delta$ at all intermediate steps between 0 and m, where one needs to finally integrate over all $x_m \geq \delta$. To compute this probability, it is convenient to change variables $y_k = x_m - x_{m-k}$, i.e., observe the sequence $\{y_k\}$ with respect to the last position and measure time backwards. Then, $r_m(\delta)$ is the probability that the new walker y_k, starting at the new origin at $k = 0$, makes a jump $\geq \delta$ at the first step and then subsequently up to m steps stays above δ, i.e.,

$$r_m(\delta) = \mathrm{Prob}\left[y_1 \geq \delta, y_2 \geq \delta, \dots, y_m \geq \delta | y_0 = 0\right]. \quad (4)$$

To compute the probability $r_m(\delta)$ in (4), we note that in the first step, the walker jumps to $y_1 = z + \delta$ from $y_0 = 0$ where $z \geq 0$ and subsequently up to $(m-1)$ steps it stays above the level δ. Writing $y_k = z_k + \delta$, we can re-express $r_m(\delta)$ as

$$r_m(\delta) = \int_0^\infty f(z+\delta)\, q_{m-1}(z) dz \quad (5)$$

where $q_n(z)$ is the probability that a random walker, starting initially at z, stays positive up to n steps. This persistence probability $q_n(z)$ has been thoroughly studied in the literature for random walks (for a review, see [32]) with arbitrary jump density $f(\eta)$ and a general expression for its Laplace transform is known as the Pollaczek-Spitzer formula [33, 34]. It states that

$$\int_0^\infty dz\, e^{-\lambda z} \sum_{n=0}^{\infty} s^n q_n(z) = \frac{1}{\lambda\sqrt{1-s}} \phi(s, \lambda) \quad (6)$$

where

$$\phi(s, \lambda) = \exp\left[-\frac{\lambda}{\pi} \int_0^\infty \frac{\ln\left(1 - s\hat{f}(k)\right)}{\lambda^2 + k^2}\, dk\right] \quad (7)$$

and $\hat{f}(k) = \int_\infty^\infty f(\eta)\, e^{i\,k\eta}\, d\eta$ is the Fourier transform of the jump density $f(\eta)$. Note that when $\delta \to 0$, the integral in (5) is just $q_m(0)$. Thus $r_m(0) = q_m(0)$. From the Pollaczek-Spitzer formula in (6), one can show [32] that $\sum_{m=0}^{\infty} q_m(0) s^m = 1/\sqrt{1-s}$, independent of the jump density. This is the celebrated Sparre Andersen theorem [35] and when inverted it simply gives $q_m(0) = \binom{2m}{m} 2^{-2m}$. When substituted back in (2), it then provides the universal result [22] in (1).

However, we are interested in the case of $\delta > 0$. To compute $r_m(\delta)$ for large m in (5), we need to know the large m behavior of $q_m(z)$ for a fixed $z > 0$. This can be extracted by analyzing (6) near $s = 1$. One finds that the leading order behavior of the right hand side of (6) near $s = 1$ is simply $[\phi(1, \lambda)/\lambda](1-s)^{-1/2}$. This means that $q_n(z)$ for large n, with fixed z, must behave like $q_n(z) \approx h(z)/\sqrt{\pi\, n}$. Substituting this on the left side of (6) and analyzing the leading behavior near $s = 1$ shows that the left hand side of (6), near $s = 1$, behaves as $\tilde{h}(\lambda)(1-s)^{-1/2}$, where $\tilde{h}(\lambda) = \int_0^\infty h(z) e^{-\lambda z} dz$ is the Laplace transform of $h(z)$. Comparing the left and right sides of (6), we obtain, for large n

$$q_n(z) \approx \frac{h(z)}{\sqrt{\pi n}} \quad \text{with} \quad \tilde{h}(\lambda) = \int_0^\infty h(z)\, e^{-\lambda z}\, dz = \frac{1}{\lambda}\phi(1,\lambda) \tag{8}$$

where $\phi(1,\lambda)$ can be read off (6) as

$$\phi(1,\lambda) = \exp\left[-\frac{\lambda}{\pi}\int_0^\infty \frac{\ln\left(1-\hat{f}(k)\right)}{\lambda^2+k^2}\, dk\right]. \tag{9}$$

Substituting the asymptotic behavior of $q_n(z)$ from (8) in (5), we obtain, for large m,

$$r_m(\delta) \approx \frac{U(\delta)}{\sqrt{\pi m}}, \quad U(\delta) = \int_0^\infty dz f(z+\delta)h(z)\,. \tag{10}$$

Finally, substituting this asymptotic behavior of the record rate $r_m(\delta)$ in Eq. (2) and performing the sum for large n, we find that the mean number of records for large n is given by

$$\langle R_n\rangle \xrightarrow[n\to\infty]{} S(\delta)\, n^{1/2}, \quad S(\delta) = \frac{2}{\sqrt{\pi}}\int_0^\infty f(z+\delta)h(z)dz\,. \tag{11}$$

This is the main exact result: for an arbitrary jump density $f(\eta)$, the mean record number grows universally as $n^{1/2}$ for large n (as in the $\delta = 0$ case), while the pre-factor $S(\delta)$ depends on δ and does so non-universally insofar as its expression depends explicitly on the jump density $f(\eta)$.

Although we have an exact expression for $S(\delta)$ for arbitrary $f(\eta)$, its explicit evaluation for all δ is difficult. For instance, to compute it explicitly for arbitrary jump density $f(\eta)$, we need to first compute its Fourier transform $\hat{f}(k)$, evaluate $\phi(1,\lambda)/\lambda$ from (9), then invert the Laplace transform (8) to obtain $h(z)$ and finally perform the integral in (11) to determine the amplitude $S(\delta)$.

For the special case of an exponential jump density, $f(\eta) = (b/2)\exp[-b\,|\eta|]$, it is possible to evaluate the pre-factor $S(\delta)$. Here, $\hat{f}(k) = b^2/(b^2+k^2)$; substituting this in the expression of $\phi(1,\lambda)$ and performing the integral yields $\phi(1,\lambda) = (b+\lambda)/\lambda$. Hence, $\tilde{h}(\lambda) = (b+\lambda)/\lambda^2$. This Laplace transform can be readily inverted to give $h(z) = 1 + bz$. Using this explicit form of $h(z)$ in the expression for $S(\delta)$ in (11) and performing the integral yields an exact expression for the pre-factor, valid for all $\delta \geq 0$

$$S(\delta) = \frac{2}{\sqrt{\pi}}\, \exp\left[-b\,\delta\right]. \tag{12}$$

Note that when $\delta \to 0$, one recovers the universal pre-factor $2/\sqrt{\pi}$.

Consider next a jump density, $f(\eta)$, whose tail decays as $f(\eta) \sim \exp[-|\eta|^a]$ for large η, where $a > 0$. Substituting this in the expression for $S(\delta)$ in (11), expanding for large δ and using $h(0) = 1$, one can show that for large δ, $S(\delta) \sim \delta^{1-a}\, e^{-\delta^a}$. For example, for the Gaussian distribution, $f(\eta) = e^{-\eta^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$, one finds that

$$S(\delta) \xrightarrow[\delta\to\infty]{} \frac{\sqrt{2}}{\pi}\,\frac{\sigma}{\delta}\, e^{-\delta^2/2\sigma^2}\,. \tag{13}$$

Finally, consider jump densities with power law tails, $f(\eta) \sim |\eta|^{-\mu-1}$ for large η with $\mu > 0$. For Lévy flights, $0 < \mu < 2$, whereas for jump densities with a finite variance, $\mu \geq 2$. The Fourier transform, $\hat{f}(k)$, for small k, generically behaves as

$$\hat{f}(k) \xrightarrow[k\to 0]{} 1 - |ak|^\mu + O(k^2). \tag{14}$$

In this case, for large δ the dominant contribution to the integral $S(\delta) = (2/\sqrt{\pi})\int_0^\infty h(z)f(z+\delta)dz$ comes from the large z region. For large z, one can show that $h(z)$ has the asymptotic behavior

$$h(z) \approx \frac{1}{a^{\mu/2}\,\Gamma[1+\mu/2]}\, z^{\mu/2} \quad \text{for } \mu < 2 \tag{15}$$

$$\approx \frac{\sqrt{2}}{\sigma}\, z \quad \text{for } \mu \geq 2. \tag{16}$$

Now, consider $S(\delta) = (2/\sqrt{\pi})\int_0^\infty f(z+\delta)\, h(z)\, dz$. We first rescale $z = \delta y$. This gives $S(\delta) = (2/\sqrt{\pi})\,\delta\int_0^\infty f(\delta(y+1))\, h(y\delta)\, dy$. For large δ, we use (16) to obtain,

$$S(\delta) \xrightarrow[\delta\to\infty]{} \sim \delta^{-\mu+\alpha} \tag{17}$$

where $\alpha = \mu/2$ for $\mu \leq 2$ and $\alpha = 1$ for $\mu \geq 2$. Thus, in this case $S(\delta)$ decays as a power law for large δ.

To test these analytical predictions we have performed Monte Carlo simulations for the three jump densities: (i) $f(\eta) = (1/2)\exp[-|\eta|]$ (Exponential); (ii) $f(\eta) = (1/\sqrt{2\pi})\exp[-\eta^2/2]$ (Gaussian), and (iii) $f(\eta)$ drawn from a Lévy distribution with Lévy exponent $\mu = 1$. The Lévy random number was generated using the method of [36] and [37]. While (i) and (ii) represent normal Fickian diffusion, the Lévy case represents non-Fickian (anomalous) diffusion; the latter can arise in diverse heterogeneous domains such as cells [38], cold atoms [39], and disordered porous media [40, 41].

Our simulations are conducted with an ensemble of independent random walkers, each entering the one-dimensional system at the origin, and with the jump length at each step drawn independently from a given pdf. In every simulation, 5000 particles take 10^6 steps each. In all cases, the particle is moved from step to step according to its actual (sampled) location, without including the value of δ; δ is added as a fixed fraction of the mean (median, for the Lévy pdf) jump length. At each step, the location of the particle is calculated and the current distance value must exceed the last maximum by at least the measurement error δ to qualify as a new δ-record and be counted; otherwise we ignore it. The simulations confirm the $n^{1/2}$ scaling for the growth of mean number of δ-records, for all values of δ. Furthermore, the three analytical predictions for $S(\delta)$ in (12), (13) and (17) are com-

pared to Monte Carlo simulations in Fig. 1; the agreement is excellent. The pre-factor $S(\delta)$ decreases from its universal value $S(0) = 2/\sqrt{\pi}$ as δ increases, so that fewer records are counted as the error increases. It is seen that the decrease in $S(\delta)$ is steepest for the Gaussian pdf and has a much slower decay for the Lévy pdf, in complete agreement with theory. The slowing down in the Lévy case is due to the anomalously skewed nature of the pdf, with frequent small jumps and rare but enormous leaps; as a consequence, potential records set by small jumps are more prone to being eliminated by the δ error. In contrast, the Gaussian case with $\sigma = 1$ displays a rapid decline with the increasing error, due to the compactness of the pdf, so that large jumps are quite rare and record events larger than the error are rarer yet.

We now proceed to examine the influence of the measurement noise γ. Let $\{x_0 = 0, x_1, x_2, \ldots, x_n\}$ represent the successive positions of the random walker. In this case, a record is registered at step m if

$$x_m + \mathcal{N}(0,\gamma)\,\Delta x > \max(0, x_0, x_1, \ldots, x_{m-1}) \qquad (18)$$

where $\mathcal{N}(0,\gamma)$ is a zero-mean Gaussian random variable with a standard deviation γ. The characteristic magnitude of the jump length, Δx, is chosen as $\Delta x = 1$ for the exponential pdf (i) and $\Delta x = \sigma = 1$ for the Gaussian pdf (ii); for the Lévy pdf (iii), Δx is the median of the one-sided Lévy distribution with $\mu = 1$. The term $\mathcal{N}(0,\gamma)\,\Delta x$ in (18) mimics the measurement noise. The noise is added for the purpose of record verification at each step and is not accumulated to the actual sequence. An analytical treatment analogous to that for δ is not yet available and we resort to numerical experiments, similar to those for δ, with the results shown in Fig. 2.

While the scaling $\langle R_n \rangle \sim T(\gamma)\, n^{1/2}$ for large n persists, in stark contrast to the $S(\delta)$, the pre-factor $T(\gamma)$ shown in Fig. 2 is an increasing function of γ for all jump densities. Thus for γ-records, the noise adds spuriously to the record-setting events, leading to false accounting of records and rendering an *apparent* $\langle R_n \rangle$ larger than the actual one. This spuriously large rate of record formation increases with the magnitude of the noise and suggests that it might be possible to infer "signal-to-noise" ratio in diffusion-type experiments by means of record counting.

For γ-records, the universality of record-setting affords the opportunity to estimate, a priori, the magnitude of γ in an experiment involving an ensemble of measurements. At least in principle, one first determines from an experiment the pdf of the jump lengths in the domain. This pdf can then be employed in random walk simulations, as shown above, to generate a curve for the pre-factor $T(\gamma)$ (such as seen in Fig. 2). Returning then to an ensemble of experimental measurements in the real system, one determines T and then reads off the corresponding value of γ from the simulated $T(\gamma)$ curve. This may provide a practical and simple algorithm to estimate the measurement noise γ in a given experimental setup.

The results presented here illustrate the subtlety and richness of record-breaking and counting, in the presence of instrumental error δ and measurement noise γ, in systems where the underlying process can be modelled by a random walk. The decoupling of the growth exponent ($1/2$, regardless of precision and noise) from the pre-factor (which depends on instrumental precision and noise in a monotonic, contrasting, and pdf-dependent manner) is significant. While the universality of the mean record number persists, $\langle R_n \rangle \sim n^{1/2}$, the magnitude of the pre-factor is sensitive to the presence of δ or γ. Such sensitivity can, perhaps, be exploited in real experiments to infer instrumental uncertainty and noise from record-counting data.

Finally, we note that all the above Monte Carlo simulations were also performed on 2d and 3d orthogonal lattices. The universality of the $n^{1/2}$ record-setting scaling is robust for all dimensions, and in all cases, the pre-factors displayed qualitative behaviors similar to those shown in Figs. 1 and 2. Moreover, Monte Carlo simulations accounting for two-sided records (absolute distance) demonstrated the same $n^{1/2}$ and similar qualitative behavior for the dependence of the pre-factors on δ and γ.

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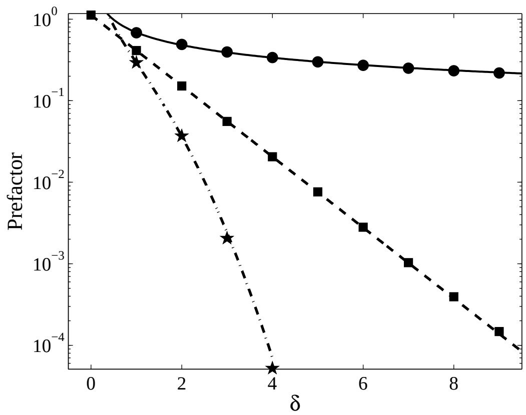

FIG. 1: One-dimensional pre-factor $S(\delta)$ versus measurement error δ with Gaussian (stars), exponential (squares; $b = 1$) and Lévy (circles; $\mu = 1$) jump length pdf's. The curves (dotted-dashed, dashed, solid) are the corresponding analytical results from (13), (12) and (17) with, respectively, functional forms $\frac{\sqrt{2}}{\pi\,\delta}\,\exp[-\delta^2/2]$, $(2/\pi^{1/2})\,\exp(-\delta)$, and $0.69\,\delta^{-0.51}$. In the Lévy case, $\mu = 1$, hence $\alpha = \mu/2 = 1/2$, and the theoretical prediction $\sim \delta^{-1/2}$ in Eq. (17) is consistent with simulations.

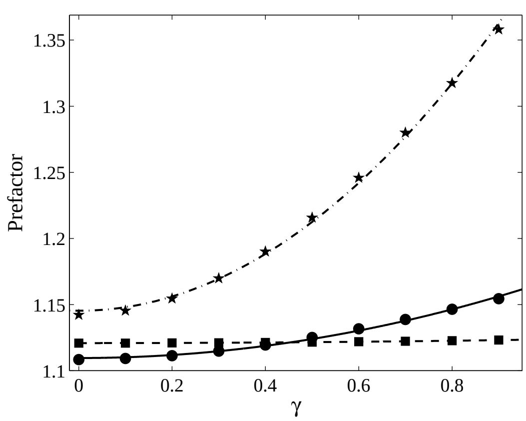

FIG. 2: Pre-factor $T(\gamma)$ as a function of the measurement noise γ for jump lengths (in one dimension) with Gaussian (stars), exponential (squares; $b = 1$) and Lévy (circles; $\mu = 1$) pdf's. The curves represent quadratic fits of functional form $a + b\gamma^2$ with different values of a, b.